

A remark on the BA model of scale-free networks

Shinji Tanimoto*

Department of Mathematics, Kochi Joshi University,
Kochi 780-8515, Japan.

Abstract

The degree distributions of many real world networks follow power-laws whose exponents tend to fall between two and three. Within the framework of the Barabási-Albert model (BA model), we explain this empirical observation by a simple fact. To that end we propose a modified BA model with one parameter that serves as a regulatory factor for the growth rate of added links in scale-free networks. The regulatory factor has something to do with the obvious fact that one link has two nodes. The modified model also allows to connect nodes by newly added links that do not necessarily emanate from new nodes. Another related model using the master equation is also given, from which the same power-law degree distribution can be derived.

1. Introduction

In recent years, it has been revealed that many real networks obey the power-law degree distribution:

$$p(k) \propto k^{-\gamma}.$$

Here $p(k)$ is the probability that a randomly chosen node (or vertex) has k links (or edges), and the exponent γ is a constant, usually

$$2 \leq \gamma \leq 3.$$

Such networks are called *scale-free* in [1]. Some examples of scale-free networks and their exponents are exhibited in the following table ([3]).

Network	γ
WWW (indegree)	2.1
WWW (outdegree)	2.72
Internet (domain)	2.1-2.2
Internet (router)	2.48
Movie actors	2.3
Phone-call	2.1

There exist also a few networks whose degree distributions follow power-laws with exponents $\gamma < 2$ or $\gamma > 3$. Traditionally, networks were thought of as random graphs, which were studied by Erdős and Rényi in the 1960s. For those the degree distribution is a binomial or Poisson distribution, which is peaked

about the average and decaying exponentially away from it.

In order to explain the above empirical fact of real networks, Barabási and Albert proposed a model (BA model) in [1]. Their model could deduce the degree distribution: $p(k) \propto k^{-3}$, *i.e.*, $\gamma = 3$.

Now we are able to find many other network models by which power-law degree distributions with a variety of exponents can be deduced (see [4, 5] or references therein). However, it seems that the BA model even now provides a constructive model of complex networks for interested researchers and students.

The objective of this paper is to produce scale-free networks with exponents between two and three based on the BA model. For that purpose we propose a modified BA model having one parameter. The parameter regulates the growth rate of added links in scale-free networks and it has something to do with the obvious fact that one link has two nodes. Another related model using the master equation ([5]) is also given, from which the same power-law degree distribution can be derived.

2. A modified BA model

In this section, reviewing the BA model ([1]) in more detail, we will propose a modification of the model. The modified BA model is involved with a parameter and it coincides with the BA model in a

*tanimoto@cc.kochi-wu.ac.jp

particular case. We derive the condition for the parameter and describe how it affects our model.

The BA model is constructed by two ingredients. First, a network grows by continuously admitting new nodes to it and new nodes connect to nodes already present in the network. Thus it expands by acquiring more nodes and links over time. Second, the manner in which a new node connects to existing nodes is not completely random. The mechanism of connection is based on so-called ‘preferential attachment’. This means that the probability that a new node connects to an existing node is proportional to its degree.

The proposed model is as follows. We start with a network having a small number of nodes, m_0 , and some links. Notice that the value m_0 and the number of the original links are not important, since we eventually take the limit with respect to time. New nodes having some new links constantly enter the network. Whenever a new node enters the network, we advance the time-step by one, and m new links are added to the network in each time-step, including the links emanating from the new node. Here m is a positive integer.

In our model, two nodes which already exist in the network are allowed to connect each other. This possibility was not considered in the BA model ([1, 2]). Connections to existing nodes are always executed by preferential attachment. Preferential attachment means that the probability, \mathcal{P} , that any node connects to a node v is proportional to the connectivity k of that node v , so that

$$\mathcal{P}(k) = \frac{k}{\sum_j k_j}. \quad (1)$$

After t time-steps, we obtain a network with $t+m_0$ nodes and mt links plus the original links. Because of the mechanism of preferential attachment, a node that happens to acquire more links than others will continue to increase its connectivity; a ‘rich-get-richer’ phenomenon.

Thus the rate at which a node v_i with k_i links acquires more links is given by

$$\frac{dk_i}{dt} = g\mathcal{P}(k_i), \quad (2)$$

where g is a proportionality constant, which is to be determined later. The denominator in (1), which we denote by

$$\mathcal{L}(t) = \sum_j k_j,$$

is the total of all degrees in the network at time-step t . In view of (1), taking the sums over i in both sides

of (2), we have

$$\frac{d\mathcal{L}(t)}{dt} = g. \quad (3)$$

Since m links are added to the network in each time-step and each link is counted twice in counting degrees, the total degree $\mathcal{L}(t)$ at time-step t satisfies

$$\mathcal{L}(t) \approx 2mt. \quad (4)$$

Therefore, in view of (3) and (4), the growth rate g of the total degree is not equal to m as in [1, 2], but approximately equal to $2m$. The growth rate will be equal to m , when one completely neglects the degrees of new nodes. On the other hand, it is equal to $2m$, when one takes a full account of them. So it appears that taking $g = m$ underestimates g .

Considering the contribution of new nodes to the growth rate g to some extent, we introduce a parameter β into the BA model by $g = \beta m$. It is reasonable to assume that β satisfies $1 \leq \beta \leq 2$ from the above argument. The parameter serves as a regulatory factor for the growing network.

Thus equation (2), together with (4) and $g = \beta m$, can be written as

$$\frac{dk_i}{dt} = \beta m \frac{k_i}{2mt} = \beta \frac{k_i}{2t}. \quad (5)$$

For the value of β we need the corresponding initial condition imposed upon a new node v_i , which enters the network at time-step t_i . This condition is the only equation that prescribes the new node, while (3) and (5) are supposed to be equations for those nodes that are already part of the network.

From (3) we see that $g = \beta m$ is the amount of degree’s increase for the part of the existing network within one time-step. On the other hand, $2m$ is the total amount of degree’s increase. Thus the initial condition is given by

$$k_i(t_i) = (2 - \beta)m, \quad (6)$$

for each new node.[¶]

Next let us examine the condition for β in further detail from (6). It necessarily implies $\beta < 2$ and $\beta \geq 1$. The former is obvious. For the latter, suppose $\beta < 1$. Then it follows from (6) that the number of new links emanating from a new node is greater than m , which is impossible. Therefore, the appropriate condition for the parameter β is

$$1 \leq \beta < 2.$$

Moreover, in the case of $1 < \beta < 2$, notice that condition (6) permits two nodes already present in the network to connect each other by some new links. The

[¶](6) is the major correction to the previous versions.

number of such links, among m newly added ones, is equal to

$$m - (2 - \beta)m = (\beta - 1)m.$$

When $\beta = 1$ as in [1, 2], the possibility is forbidden. That is, in this case all new links are connections between only a new node and nodes of the existing network.

In Section 3, under the condition $1 \leq \beta < 2$, we show that this modified BA model, (5) together with (6), generates a scale-free network with exponent

$$\gamma = 1 + \frac{2}{\beta}.$$

So we have $2 < \gamma \leq 3$ for our model. In Section 4, using a quite different approach as in [5], a similar power-law will be obtained by means of the master equation based on (5), too.

3. Scale-free networks with exponents between 2 and 3

Under the assumption $1 \leq \beta < 2$, the solution of (5) with initial condition (6) is given by

$$k_i(t) = (2 - \beta)m \left(\frac{t}{t_i} \right)^{\beta/2} \quad (t \geq t_i),$$

where t_i is the time when a node v_i enters the network. It is assumed that the time interval $s = t_{i+1} - t_i$ (for each i) is a constant, usually $s = 1$.

First we verify condition (4) for the model. The total degree $\mathcal{L}(t)$ equals

$$(2 - \beta)m \left[\left(\frac{t}{t_1} \right)^{\beta/2} + \left(\frac{t}{t_2} \right)^{\beta/2} + \cdots + \left(\frac{t}{t_i} \right)^{\beta/2} \right],$$

where t_i is the maximum time-step such that $t_i \leq t$. The sum in the square brackets is approximately equal to $2(2 - \beta)^{-1}t$ by the continuous approximation:

$$\int_{t_1}^t \left(\frac{t}{x} \right)^{\beta/2} dx.$$

Hence we see that (4) is satisfied.

In order to obtain the degree distribution for this model, we could adopt the method employed in [2]. However, we will use the following alternative approach, because it is easier to follow and, moreover, it gives the same asymptotic behavior for the degree distribution as the method in [2].

For the probability $p(k)$ that a randomly chosen node v has a connectivity k , it suffices to estimate the

number of time-steps t_i 's for nodes v_i 's, which satisfy the inequalities

$$k - 1 < k_i(t) = (2 - \beta)m \left(\frac{t}{t_i} \right)^{\beta/2} \leq k, \quad (7)$$

or alternatively

$$k - 1/2 < k_i(t) \leq k + 1/2.$$

Let us take (7) for the estimation and rewrite it as

$$t \left[\frac{k}{(2 - \beta)m} \right]^{-2/\beta} \leq t_i < t \left[\frac{k - 1}{(2 - \beta)m} \right]^{-2/\beta}.$$

In the case of $s = 1$, the number of nodes satisfying (7) is equal to the difference of both sides of the above inequalities:

$$t\{(2 - \beta)m\}^{2/\beta}\{(k - 1)^{-2/\beta} - k^{-2/\beta}\},$$

or

$$t\{(2 - \beta)m\}^{2/\beta}\{(1 - 1/k)^{-2/\beta} - 1\}k^{-2/\beta}.$$

Making use of $(1 - 1/k)^{-2/\beta} \approx 1 + 2\beta^{-1}/k$ for large k , we see that the number of nodes satisfying (7) is approximated by

$$2\beta^{-1}t\{(2 - \beta)m\}^{2/\beta}k^{-1-2/\beta}.$$

Since at time t there exist $t + m_0$ nodes, the probability $p(k)$ is thus given by

$$\frac{2\beta^{-1}t\{(2 - \beta)m\}^{2/\beta}}{t + m_0}k^{-1-2/\beta}.$$

In the limit $t \rightarrow \infty$ we conclude

$$p(k) \propto k^{-(1+2/\beta)},$$

i.e., $\gamma = 1 + 2/\beta$, for all large k .

4. Another related model

In this section we assume $1 \leq \beta \leq 2$. Related to the model discussed in Section 2, we propose another by means of the master equation as in [5]. Making use of (5), we show that it also leads us to the same power-law.

Let $p(k, t_i, t)$ be the probability that a node v_i , entering the network at time-step t_i , has a degree k at time t . The master equation for it is given by

$$\begin{aligned} p(k, t_i, t + 1) &= \beta \frac{k - 1}{2t} p(k - 1, t_i, t) \\ &+ \left(1 - \beta \frac{k}{2t} \right) p(k, t_i, t), \end{aligned} \quad (8)$$

because the degree of a node with k links increases by one at the rate $\beta k/2t$ from (5).

We define the limiting probability

$$p(k) = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{t_i} p(k, t_i, t)$$

for deriving the degree distribution. It follows from (8) that the recurrence relation for this $p(k)$ is given by

$$(t+1)p(k) = \beta \frac{k-1}{2t} t p(k-1) + \left(1 - \beta \frac{k}{2t}\right) t p(k),$$

or

$$p(k) = \frac{k-1}{k+2/\beta} p(k-1).$$

Using well-known properties of the beta function and the gamma function, it can be written as

$$\begin{aligned} p(k) &= \frac{\Gamma(k)\Gamma(1+2/\beta)}{\Gamma(k+1+2/\beta)} p(1) \\ &= B(k, 1+2/\beta) p(1) \\ &\propto k^{-(1+2/\beta)}, \end{aligned}$$

for all large k . The last asymptotic behavior follows from Stirling's formula.

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